Q1. *[Warm-up.]* Express the following complex numbers in the form $a + bi$, with $a, b \in \mathbb{R}$.

(a) $(4 - 5i) + (1 + 2i)$.
(b) $(3 + 5i)(1 - i)$.
(c) $(2 + 2i)^2$.
(d) i^5.

Solution:

(a) $(4 - 5i) + (1 + 2i) = (4 + 1) + (-5 + 2)i = 5 - 3i$.
(b) $(3 + 5i)(1 - i) = 3 - 3i + 5i - 5i^2 = 3 + 2i + 5 = 8 + 2i$.
(c) $(2 + 2i)^2 = 2^2 + 2(2)(2i) + (2i)^2 = 4 + 8i + 4i^2 = 8i = 0 + 8i$.
(d) $i^5 = i^2 i^2 i = (-1)(-1)i = i = 0 + (1)i$.

Q2. *[Trigonometric form.]* Express the following complex numbers in trigonometric form

$$z = r(\cos\theta + i\sin\theta).$$

(a) i.

(b) $-i$.

(c) 3. (Remember: real numbers *are* compelx numbers – they just have zero imaginary part!)

(d) -3.

(e) $1 + i$.

(f) $3 + 4i$.

Solution:

(a) $z = (1)(\cos(\pi/2) + i\sin(\pi/2))$.

(b) $z = (1)(\cos(3\pi/2) + i\sin(3\pi/2))$.

(c) $z = (3)(\cos(0) + i\sin(0))$.

(d) $z = 3(\cos(\pi) + i\sin(\pi))$.

(e) $z = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$.

(f) $z = 5(\cos(\theta) + i\sin(\theta))$, where $\theta = \tan^{-1}(4/3) \approx 0.927$ rad.

Q3. *[Polar form.]* Express the complex numbers in **Q2** in polar form

$$z = re^{i\theta}.$$

Solution:

(a) $z = (1)e^{i\pi/2}$.

(b) $z = (1)e^{i(3\pi)/2}$.

(c) $z = (3)e^{i0}$.

(d) $z = 3e^{i\pi}$.

(e) $z = \sqrt{2}e^{i(\pi/4)}$.

(f) $z = 5e^{i\theta}$, where $\theta = \tan^{-1}(4/3) \approx 0.927$ rad.

Q4. *[De Moivre's Theorem.]* Using mathematical induction, prove that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

for all positive integers n.

Note: The result is also true for negative integers (and zero). Can you prove this too?

Solution:

We will proceed by mathematical induction. The result is clearly true when $n = 1$. Let's assume that the result is true for $n = k$, where $k \geq 1$ is an integer, and let's consider what happens when $n = k + 1$. We have

$$\begin{aligned}
[r(\cos\theta + i\sin\theta)]^{k+1} &= [r(\cos\theta + i\sin\theta)]^k \; r(\cos\theta + i\sin\theta) \\
&= r^k(\cos k\theta + i\sin k\theta) \; r(\cos\theta + i\sin\theta) \qquad \text{(by assumption for } n = k\text{)} \\
&= r^{k+1}[(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i\,(\cos k\theta\sin\theta + \cos\theta\sin k\theta)] \\
&= r^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta)) \qquad \text{(trig identities!)} \\
&= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta).
\end{aligned}$$

This shows that if the identity is true if $n = k$ then it must be true for $n = k + 1$. Hence, since the identity is true for $n = 1$, it must be the case that the identiy is tru for $n = 2, 3, \ldots$. Therefore, by the principle of mathematical induction, the identity is true for all positive integers n.

The identity is also true if $n = 0$, because the left side in this case is

$$[r(\cos\theta + i\sin\theta)]^0 = 1$$

and the right side is also

$$r^0(\cos 0 + i\sin 0) = (1)(1 + 0i) = 1.$$

If n is negative integer, say $n = -k$ with $k > 0$, we want to prove that

$$[r(\cos\theta + i\sin\theta)]^{-k} = r^{-k}(\cos(-k\theta) + i\sin(-k\theta)).$$

By multiplying both sides by $[r(\cos\theta + i\sin\theta)]^k$, this equation is the same as

$$1 = r^{-k}[r(\cos\theta + i\sin\theta)]^k(\cos(-k\theta) + i\sin(-k\theta)).$$

Let's try to prove this, by starting with the right-side and showing that it must be equal to 1. We have

$$\begin{aligned}
&r^{-k}[r(\cos\theta+i\sin\theta)]^k(\cos(-k\theta)+i\sin(-k\theta))\\
&\qquad = r^{-k}r^k(\cos k\theta + i\sin k\theta)(\cos(-k\theta)+i\sin(-k\theta)) && \text{(by DeMoivre for } k>0\text{)}\\
&\qquad = \cos k\theta\cos(-k\theta) - \sin k\theta\sin(-k\theta) + i(\cos k\theta\sin(-k\theta) + \cos(-k\theta)\sin k\theta)\\
&\qquad = \cos(k\theta - k\theta) + i\sin(k\theta - k\theta) && \text{(trig identities!)}\\
&\qquad = \cos 0 + i\sin 0\\
&\qquad = 1 + 0i\\
&\qquad = 1,
\end{aligned}$$

as required! This completes the proof.

Q5. *[nth roots of complex numbers.]* Using De Moivre's theorem, find the following roots. Express your answers in both standard and trigonometric forms.

Remember – the strategy is as follows. If we have a complex number, like $w = 4i$, and we want to find its 5th roots say, then we are looking for all complex numbers z that satisfy $z^5 = w$.

(a) The square roots of i.

(b) The third roots of 8.

(c) The fourth roots of 1. (See also **Q6**.)

(d) The fifth roots of $1 + i$.

Solution:

(a) We know that $i = (1)(\cos(\pi/2) + i\sin(\pi/2))$. So if $z = r(\cos\theta + i\sin\theta)$ is a square root of i (meaning $z^2 = i$), we must have

$$r^2 = 1 \quad \text{and} \quad 2\theta = \frac{\pi}{2} + 2k\pi.$$

This gives us $r = 1$ (since r must be positive) and $\theta = \frac{\pi}{4} + k\pi$. That is,

$$z = \cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right).$$

Now, because cos and sin are 2π-periodic, we only need to consider $k = 0$ and $k = 1$ (the values of k will not give us anything new!). So the square roots of i are:

- When $k = 0$: $z = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.
- When $k = 1$: $z = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

This seems a bit weird! Let's confirm:

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} + 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}i + \frac{1}{2}i^2 = \frac{1}{2} + i - \frac{1}{2} = i.$$

So the first one is correct! The second one is too, by a similar check.

(b) We know that $8 = (1)(\cos(0) + i\sin(0))$. So if $z = r(\cos\theta + i\sin\theta)$ is a tird root of 8 (meaning $z^3 = 8$), we must have

$$r^3 = 8 \quad \text{and} \quad 3\theta = 0 + 2k\pi.$$

Proceeding as in part (a), we get the 3 third roots

$$\sqrt[3]{8}, \quad \sqrt[3]{8}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \quad \text{and} \quad \sqrt[3]{8}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

We can simplify the latter two into

$$\sqrt[3]{8}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \quad \text{and} \quad \sqrt[3]{8}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right).$$

At this point, it would be an instructive exercise to cube each of the above to confirm that we do indeed get 8. I'll leave it to you to do this!

(c) We know that $1 = (1)(\cos(0) + i\sin(0))$. So if $z = r(\cos\theta + i\sin\theta)$ is a fourth root of 8 (meaning $z^4 = 8$), we must have

$$r^4 = 1 \quad \text{and} \quad 4\theta = 0 + 2k\pi.$$

Proceeding as in parts (a) and (b), we get the 4 fourth roots

$$1, \quad \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}, \quad \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4} \quad \text{and} \quad \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}.$$

We can simplify these into:

$$1, \quad i, \quad -1, \quad -i.$$

And indeed, we can check that the fourth power of each of these is equal to 1. For instance,

$$(-i)^4 = (-i)(-i)(-i)(-i) = (-1)^4 i^2 i^2 = (1)(-1)(-1) = 1.$$

Q6. *[Roots of unity.]* The complex solutions to the equation $z^n = 1$ are called the nth roots of unity. There are n distinct nth roots of unity, for each positive integer n. Let

$$\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right).$$

(a) Show that ω is an nth root of unity. [**Hint:** What is ω^n?]

(b) Show that $\omega^2,\ \omega^3,\ \ldots, \omega^{n-1}$ are also nth roots of unity and that no two of them are equal. So, together with ω, these are all nth roots of unity.

(c) Determine
$$\omega + \omega^2 + \cdots + \omega^{n-1} \quad \text{and} \quad \omega\omega^2 \cdots \omega^{n-1}.$$

Solution:

(a) By De Moivre's theorem, we have

$$\begin{aligned}\omega^n &= \cos\left(\frac{2\pi n}{n}\right) + i\sin\left(\frac{2\pi n}{n}\right)\\ &= \cos(2\pi) + i\sin(2\pi)\\ &= 1 + 0i\\ &= 1.\end{aligned}$$

So $z = \omega$ satisfies the equation $z^n = 1$, meaning that ω is an nth root of unity.

(b) Notice that for *any* integer k, we have

$$(\omega^k)^n = \omega^{kn} = (\omega^n)^k = 1^k = 1.$$

So ω^k is an nth root of unity. (*Note:* In the first and second equal signs above, we're secretly using De Moivre's theorem to manipulate the exponents! That is, the laws of exponentiation that you're familiar with from the world of real numbers are still true here, provided we raise to powers of integers.)
In particular, $\omega^2,\ \omega^3,\ \ldots, \omega^{n-1}$ are all roots of unity.
Suppose two of them were equal, say $\omega^k = \omega^l$ with k and l integers between 2 and $n-1$, inclusive. Then

$$\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = \cos\left(\frac{2\pi l}{n}\right) + i\sin\left(\frac{2\pi l}{n}\right).$$

If we equate the θ's on both sides of these trigonometric forms, we conclude that $\frac{2\pi k}{n}$ and $\frac{2\pi l}{n}$ must differ by an integer multiple of 2π. Let's say

$$\frac{2\pi k}{n} = \frac{2\pi l}{n} + 2m\pi \quad (m \in \mathbb{Z}).$$

Then if we multiply through by $\frac{n}{2\pi}$, we get

$$k = l + nm.$$

But since k and l are strictly between 1 and n, they cannot differ by an integer multiple of n unless that integer multiple is 0, in which case $k = l$. This shows that if $\omega^k = \omega^l$, with k and l as above, then we must have that $k = l$. Consequently, no two of $\omega^2,\ \omega^3,\ \ldots, \omega^{n-1}$ are equal.

Q7. Let $z(t) = \cos 2\pi t + i \sin 2\pi t$, where t is a real number such that $0 \leq t \leq 1$.

(a) Imagining $z(t)$ as describing the position of a complex number in the complex plane at time t. At $t = 0$, the complex number is $z(0) = 1$ and as t increases, $z(t)$ moves in a path that ends back at $z(1) = 1$ at $t = 1$. Plot this path.

(b) Express the complex number $z(t)$ in polar form as $z(t) = r(t)e^{i\theta(t)}$. How does $\theta(t)$ change as t goes from $t = 0$ to $t = 1$?

(c) Looking at part (a), we know that $z(0) = 1$ and $z(1) = 1$, so they should have the same polar form. Does this contradict what you found in part (b)?

Solution:

(a) We get the unit circle!

(b) $\theta(t)$ goes from 0 to 2π. In particular, $\theta(t)$ changes.

(c) Although $\theta(t)$ did change we we went around the loop, the complex numbers $z(0)$ and $z(t)$ didn't, because

$$z(0) = (1)e^{i\theta(0)} = e^{i0} = \cos 0 + i \sin 0 = 1$$

and

$$z(2\pi) = (1)e^{i\theta(2\pi)} = (1)e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1.$$

Q8. Repeat **Q7** but now let t go from 0 to 2. What do you notice? What if you let t go from 0 to 3?

Solution: The same thing happens. Even though θ changes, we end up at the same complex number, because cos and sin are 2π-periodic. So as long as we do full loops around the circle, we get back the same complex number, as expected.

Q9. Going back to **Q7**(b), pick one of the two square roots of $z(t)$ using your chosen polar form. Let's call it $\sqrt{z(t)}$.

(a) What are $\sqrt{z(0)}$ and $\sqrt{z(1)}$?

(b) Recalling that $z(0) = z(1) = 1$, what do your finding say about the usage of the notation $\sqrt{z}$ for square roots of complex numbers?

Solution:

(a) Our polar form is $z(t) = e^{2\pi ti}$. So a possible square root has one-half of the $\theta(t) = 2\pi ti$ that we've chosen. That is, we can take

$$\sqrt{z(t)} = e^{\pi ti}.$$

In this case, we have $\sqrt{z(0)} = e^0 = 1$ and $\sqrt{z(1)} = e^{\pi i} = -1$.

(b) This is a bit weird! Above we got $\sqrt{z(0)} = 1$ and $\sqrt{z(1)} = -1$. But $z(0) = z(1) = 1$, so we seem to be saying that $\sqrt{1} = 1$ and $\sqrt{1} = -1$ simultaneously! So our usage of the symbol $\sqrt{\ }$ is problematic. We are not getting a well-defined function that produces a specific output for a given input. What we're getting looks like a type of "multivalued" function, which is pretty confusing! How do we choose whether $\sqrt{1} = 1$ or $\sqrt{1} = -1$?

The upshot here is that the notation $\sqrt{z}$ can be very ambiguous when we work with complex numbers. (Even if we're taking square roots of real numbers – something as simple as $z = 1$!)